Defense Technical Information Center Compilation Part Notice

ADP010919

TITLE: Pattern Selection: Nonsingular Saffman-Taylor Finger and its Dynamic Evolution with Zero Surface Tension

DISTRIBUTION: Approved for public release, distribution unlimited

This paper is part of the following report:

TITLE: Paradigms of Complexity. Fractals and Structures in the Sciences

To order the complete compilation report, use: ADA392358

The component part is provided here to allow users access to individually authored sections of proceedings, annals, symposia, ect. However, the component should be considered within the context of the overall compilation report and not as a stand-alone technical report.

The following component part numbers comprise the compilation report:
ADP010895 thru ADP010929

PATTERN SELECTION: NONSINGULAR SAFFMAN-TAYLOR FINGER AND ITS DYNAMIC EVOLUTION WITH ZERO SURFACE TENSION

HUIDAN YU

Department of Physics, Zhejiang Normal University, Jinhua321004, Zhejiang, China and Laboratory for Nonlinear Mechanics of Continuous Media, Institute of Mechanics, Chinese Academy of Sciences, Beijing, China, and Department of Physics, Peking University, Beijing 100871, China

KAIHUA ZHAO

Department of Physics, Peking University, Beijing 100871, China

By a slight modification of Saffman-Taylor's viscous finger, we remove its singularity and present a finite-length nonsingular Saffman-Taylor finger. Its dynamic evolution solutions, both one-step theory prediction and numerical simulation in the long-time limit with zero surface tension, give an alternative proof for the analytical selection demonstration of the Saffman-Taylor finger width in the absence of surface tension. As a simple example of pattern selection, which has real experimental background, this work not only contradicts the generally accepted belief that surface tension is indispensable for the selection of the $\frac{1}{2}$-width finger but also provides more models to compute the evolution, competition and ramification of multiple fingers numerically in straight channel as well as circular disc geometry.

1 Introduction

The Saffman-Taylor problem [1] has played a central role in the study of viscous fingering in a Hele-Shaw cell [2], which was modeled by two-dimensional potential flow at the interface between two fluids. Saffman and Taylor found analytically a continuous family of steady-state solutions, which shows fingers of different width could exist in the absence of surface tension, but their experiments with negligible surface-tension effect in 1958 and numerical calculations made by McLean, Saffman and Vanden-Broeck in the presence of surface-tension effect in 1981 and 1983 showed no hint of the continuous family [3,4]. On the contrary, the finger with relative width $\lambda = 0.5$ is always singled out at the zero surface tension limit. This problem (which is so called Saffman-Taylor) has been of much subsequent interest because it is universal, i.e., the same selection phenomenon is common for displacement of various viscous liquids by less viscous ones for immiscible incompressible liquids. Much work [5] was done toward solving this Saffman-Taylor puzzle. However, it has been widely accepted that the inclusion of surface tension is the only way to select the most stable finger width since Saffman-Taylor proposed that surface tension between the two fluids would solve the selection problem [1]. Although several works [6] confirmed numerical evidence of the discrete spectrum of λ, decreasing to $\frac{1}{2}$, in the limit of low surface tension, exact solutions are rare with zero surface tension. Recently, Mineev-Weinstein [7] analytically solved the finger selection problem in the absence of surface tension. He showed an exact result that a generic interface in a Hele-shaw cell evolves to nonlinearly stable single uniformly advancing finger occupying one-half of the channel width, which contradicts the generally accepted

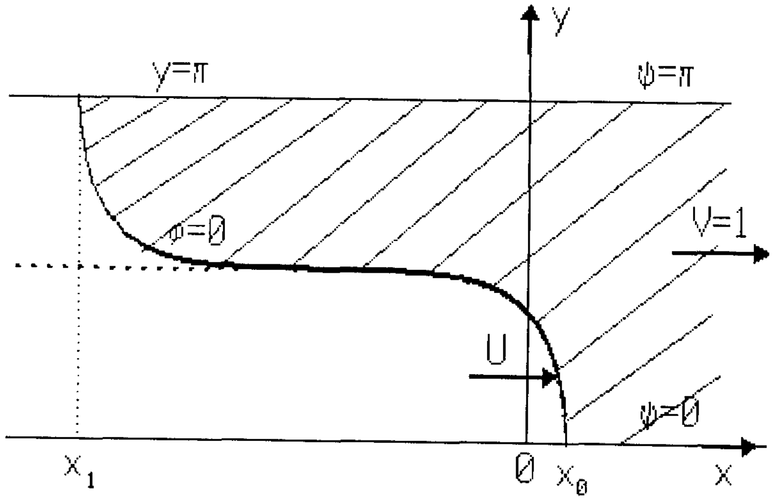

Figure 1.

belief that surface tension is indispensable for the selection of the $\frac{1}{2}$-width finger.

We find that the problem of the finger width selection stems from the singularity of Saffman-Taylor finger, which is half infinite in length. The singularity makes the numerical calculation impossible and is not in agreement with the experimental results. We introduce a positive variable $\epsilon(t)$ to remove the singularity. By the conformal mapping method presented by Bensimon [8], we predict that there exists a unique steady width $\lambda = 0.5$ of the viscous finger to which the fingers of other width approach in the absence of surface tension after a one-step analysis. Then we verify this prediction by numerical computation and show its long time evolution.

2 Nonsingular Saffman-Taylor Finger and One-step Dynamic Evolution Theory

Following Saffman and Taylor we consider the analysis of a long air bubble or finger moving through a channel in the Hele-Shaw cell bounded by two straight parallel walls. We take the y-axis (Fig. 1) perpendicular to the walls and the x-axis along the central axis of symmetry of the channel. The separation of the walls is normalized to 2π and the velocity of the fluid at infinity in front of the finger is normalized to unity. Let $z = x + iy$ and $\Phi = \phi + i\psi$, $\phi(z)$ and $\psi(z)$ being the velocity potential and stream function, respectively.

In case of the absence of surface tension, the interface is described by $\phi = 0$, which can be conformally mapped to the circumference of a unit disk in the complex plane $\zeta = e^{-\Phi}$. By these notations the Saffman-Taylor solution [1] can be written as

$$z = 2(1-\lambda)ln\frac{1}{2}(e^{-i\psi}+1)+i\psi, \tag{1}$$

where the single parameter λ depicts the width of the finger. Saffman-Taylor finger is half infinite in length and the solution is steady in the frame co-moving with the finger tip which has a speed $\frac{1}{\lambda}$ relative to the laboratory frame.

In order to remove the singularity at $\psi = \pm\pi$ in Eq. (1), we choose the solution in the following form in the laboratory frame:

$$z = 2(1-\lambda)ln\frac{1}{2}(\zeta+1+\epsilon) - ln\zeta + \frac{t}{\lambda} + \widehat{z}(\zeta,t), \tag{2}$$

where $\epsilon = \epsilon(t)(=\epsilon_0$ at t=0) is a small positive real number varying with time t, $\widehat{z}(\zeta,t)(=0$ at t=0) describes the possible additional time-dependence of the finger. The functional form of $\epsilon(t)$ and $\widehat{z}(\zeta,t)$ are to be determined by the evolution equation of interface. In Eq. (2) the singularity of Saffman-Taylor solution at $\zeta=-1$ on the unit circle is moved to outside of the unit disk and therefore the length of the finger becomes finite (Fig.1).

The corresponding equation of the boundary of the nonsingular Saffman-Taylor finger is

$$z = 2(1-\lambda)ln\frac{1}{2}(e^{-i\psi}+1+\epsilon) + i\psi + \frac{t}{\lambda} + \widehat{z}(\psi,t) \tag{3}$$

At moment t, the finger tip is located at $\psi = 0, \phi = 0$ and the rear end at $\psi = \pm\pi, \phi = 0$, therefore the length of the finger is

$$L(t) = x_0 - x_1 = 2(1-\lambda)ln(1+\frac{2}{\epsilon}) \tag{4}$$

In our notation Bensimon's equation [8] in case of the absence of surface tension is written as

$$\frac{\partial z}{\partial t} = -\zeta\frac{\partial z}{\partial\psi}G(\zeta), \tag{5}$$

where

$$G(\zeta) = A\{g(\psi)\}, \quad g(\psi) = [\frac{1}{|\zeta\partial_\zeta zG(\zeta)|^2}|]_\Gamma, \tag{6}$$

and the suffix Γ means that the value of the function in the square parentheses is to be taken on the unit circle where $\zeta = e^{-i\psi}$, A is an operator analytically continuing the real function g(ψ) defined on the circumference of the unit circle as the real part of complex function $G(\zeta)$ in the interior of it. The analytical continuation is performed by Fourier expansion.
When t=0, Eq. (2) becomes

$$z = 2(1-\lambda)ln\frac{1}{2}(\zeta+1+\epsilon_0) - ln\zeta \tag{7.1}$$

and the equation of the nonsingular Saffman-Taylor finger is

$$z = 2(1-\lambda)ln\frac{1}{2}(e^{-i\psi}+1+\epsilon_0) + i\psi \tag{7.2}$$

Then

$$g(\psi) = \frac{1 + 2(1+\epsilon_0)cos\psi + (1+\epsilon_0)^2}{(1-2\lambda)^2 - 2(1-2\lambda)(1+\epsilon_0)cos\psi + (1+\epsilon_0)^2} \tag{8}$$

taking a trial solution as

$$G(\zeta) = \frac{a + b\zeta}{(1-2\lambda)\zeta - (1+\epsilon_0)} \tag{9}$$

with a, b to be decided , we have

$$ReG(\zeta)|_\Gamma = \frac{(b+a)(1-2\lambda) - b(1+\epsilon_0)cos\psi - a(1+\epsilon_0)}{(1-2\lambda)^2 - 2(1-2\lambda)(1+\epsilon_0)cos\psi + (1+\epsilon_0)^2}. \tag{10}$$

Equating

$$ReG(\psi)|_\Gamma = g(\psi) \tag{11}$$

we get

$$b = \frac{(1-2\lambda) + (1-2\lambda)(1+\epsilon_0)^2 + 2(1+\epsilon_0)^2}{(1-2\lambda)^2 - (1+\epsilon_0)^2} \tag{12}$$

$$a = \frac{(2(1-2\lambda) + (1+\epsilon_0)^2 + 1)(1+\epsilon_0)}{(1-2\lambda)^2 - (1+\epsilon_0)^2}. \tag{13}$$

Substituting eqs.(12) and (13) in $G(\zeta)$, we get

$$-\zeta\frac{\partial z}{\partial \zeta}G(\zeta) = \frac{1}{\lambda} - \frac{(1-\lambda)(1+\epsilon_0)[1-(1+\epsilon_0)^2]}{\lambda[(1-2\lambda)^2 - (1+\epsilon_0)^2](\zeta + 1 + \epsilon_0)} -$$

$$\frac{(1-2\lambda)(1-\lambda)[1-(1+\epsilon_0)^2]\zeta}{\lambda[(1-2\lambda)^2 - (1+\epsilon_0)^2](\zeta + 1 + \epsilon_0)}. \tag{14}$$

Meanwhile, from Eq.(2) at time t, we have

$$\frac{\partial z}{\partial t} = \frac{1}{\lambda} + \frac{2(1-\lambda)}{\zeta + 1 + \epsilon}\frac{d\epsilon_0}{dt} + \frac{\partial \widehat{z}}{\partial t} \tag{15}$$

Comparing Eq.(14) and Eq.(15), we have

$$\frac{2(1-\lambda)}{\zeta + 1 + \epsilon_0}\frac{d\epsilon_0}{dt} = -\frac{(1-\lambda)(1+\epsilon_0)[1-(1+\epsilon_0)^2]}{\lambda[(1-2\lambda)^2 - (1+\epsilon_0)^2](\zeta + 1 + \epsilon_0)} \tag{16}$$

$$\frac{\partial \widehat{z}}{\partial t} = -\frac{(1-2\lambda)(1-\lambda)[1-(1+\epsilon_0)^2]\zeta}{\lambda[(1-2\lambda)^2 - (1+\epsilon_0)^2](\zeta + 1 + \epsilon_0)} \tag{17}$$

and at the boundary

$$\frac{\partial \widehat{z}}{\partial t}|_\Gamma = -\frac{(1-2\lambda)(1-\lambda)[1-(1+\epsilon_0)^2]\{[1+(1+\epsilon_0)cos\psi] - i(1+\epsilon_0)sin\psi\}}{\lambda[(1-2\lambda)^2 - (1+\epsilon_0)^2][1 + 2(1+\epsilon_0)cos\psi + (1+\epsilon_0)^2]}. \tag{18}$$

In the first time step we have

$$\Delta\widehat{z} = \frac{\partial \widehat{z}}{\partial t}\Delta t \tag{19}$$

thus

$$\Delta\widehat{x} = Re\frac{\partial \widehat{z}}{\partial t}\Delta t = -\frac{(1-2\lambda)(1-\lambda)[1-(1+\epsilon_0)^2][1+(1+\epsilon_0)cos\psi]}{\lambda[(1-2\lambda)^2-(1+\epsilon_0)^2][1+2(1+\epsilon_0)cos\psi+(1+\epsilon_0)^2]} \quad (20)$$

$$\Delta\widehat{y} = Im\frac{\partial \widehat{z}}{\partial t}\Delta t\frac{(1-2\lambda)(1-\lambda)[1-(1+\epsilon_0)^2](1+\epsilon_0)sin\psi}{\lambda[(1-2\lambda)^2-(1+\epsilon_0)^2][1+2(1+\epsilon_0)cos\psi+(1+\epsilon_0)^2]} \quad (21)$$

Choosing a suitable ϵ_0 from Eqs.(20) and (21), we can see

- If originally $\lambda > 0.5$, then $\Delta\widehat{x} > 0, \Delta\widehat{y} < 0$, the finger is being stretched to be longer and thinner in later time, i.e., λ tends to decrease.
- If originally $\lambda < 0.5$, then $\Delta\widehat{x} < 0, \Delta\widehat{y} > 0$, the finger is being compressed to be shorter and wider in later time, i.e., λ tends to increase.
- If originally $\lambda = 0.5$,then $\Delta\widehat{x} = 0, \Delta\widehat{y} = 0$, the finger keeps its width forever.

Thus in our one-step dynamic evolution theory about the evolution of nonsingular Saffman-Taylor finger, we conclude: *There exists a unique steady width $\lambda = 0.5$ of the viscous finger to which the fingers of other width approach.* This is exactly what Saffman and Taylor observed in the experiment in 1958.

3 Numerical Computation

From the one-step analysis described above we can not predict completely the shape change in the long run, however, the evolution may be obtained by numerical computation. At time t, if $z(\psi,t) = x(\psi,t) + iy(\psi,t)$ is given, considering $\zeta = e^{-i\psi}$ we have

$$[\zeta\partial_\zeta z]_\Gamma = i\partial_\psi z \quad (22)$$

substitution of eq.(22) in eq.(6) yields

$$g(\psi) = \frac{1}{(\partial_\psi y)^2 + (\partial_\psi x)^2} \quad (23)$$

According to the Poisson integral formula[9], if

$$g(\psi) = a_0 + \sum_{n=1}^{\infty}(a_n e^{in\psi} + a_n^* e^{-in\psi}) \quad (24)$$

we have

$$G(\zeta) = A\{g(\psi)\} = a_0 + 2\sum_{n=1}^{\infty}(a_n\zeta^n) \quad (25)$$

Noticing the symmetry of the finger and substituting Eqs.(22) and (23) into Bensimon equation (5), we get the nonlinear partial differential equation

$$\partial_t z = -[2\sum_{n=1}^{\infty}(a_n sinn\psi) + ig(\psi)]\partial_\psi z \quad (26)$$

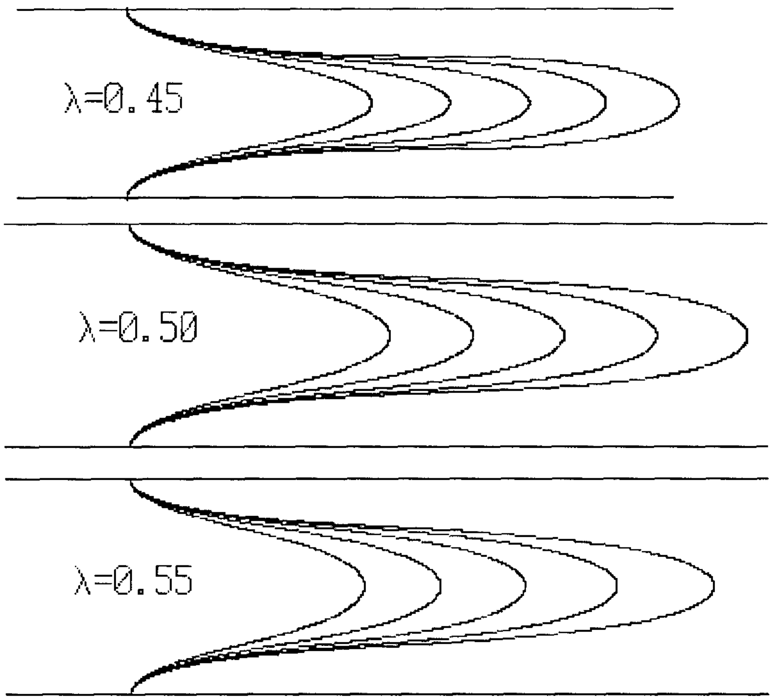

Figure 2. Stable evolution of NST with different $\lambda's, \epsilon_0 = 0.3$

where

$$a_n = \frac{1}{\pi} \int_0^{\psi} \frac{g(\psi)}{2} cosn\psi d\psi$$

If the initial condition $z(\psi, 0) = x(\psi, 0) + iy(\psi, 0)$ is given, we can compute the equation (26) with various λ and get a family of evolution. Fig. 2 shows the evolution of nonsingular Saffman-Taylor finger. It can be seen from the figure that the above conclusions are valid in the long run of evolution.

4 Conclusions and Prospect

Using the conformal mapping method presented by Bensimon, we present a nonsingular Saffman-Taylor finger. Both theoretical and numerical solutions verify the pattern selection: the tracing finger always tends to $\frac{1}{2}$-width finger in the long run with zero absence surface tension which resolves the Saffman-Taylor's puzzle. Being a simple example of pattern selection which has real experimental background, this work contradicts the generally accepted belief that surface tension is indispensable for the selection of the $\frac{1}{2}$-width finger, a result in parallel to paper in ref [7].

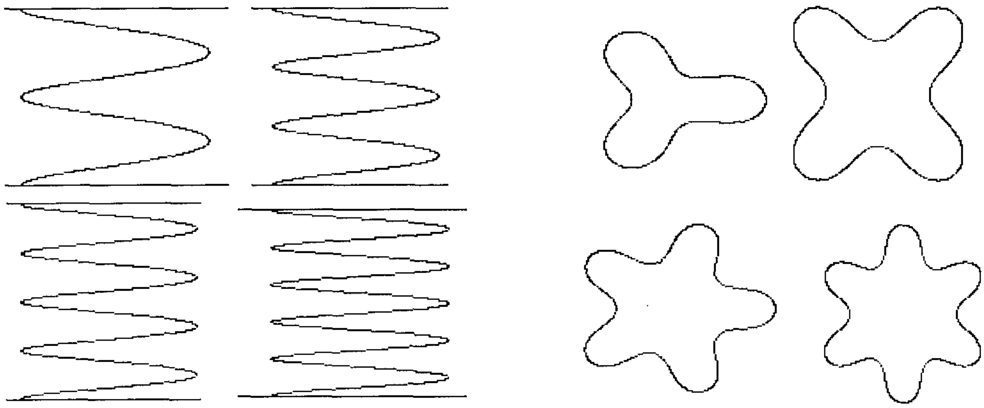

Figure 3. Straight multiple fingers (left) and round fingers (right)

Futhermore, the solution (2) of single finite finger can be generalized to cases of multiple fingers

$$z = \frac{2(1-\lambda)}{n} ln\frac{1}{2}[e^{-n\Phi+i(n-1)\pi} + 1 + \epsilon] + \Phi \tag{28}$$

and of round fingers

$$lnz = \frac{2(1-\lambda)}{n} ln\frac{1}{2}[e^{-n\Phi+i(n-1)\pi} + 1 + \epsilon] + \Phi \tag{29}$$

(see Figs. 3 and 4 respectively), may produce more interesting pattern selection problems of evolution, competition and ramification of multiple fingers in straight channel as well as circular disc geometry.

It should be mentioned that Bensimon's equation is a highly nonlinear equation which meets computational difficulty. Tiny round-off errors, especially the residual errors in summing up the Fourier series of Eq.(26) will be magnified quickly. Special efforts has been made to reduce these errors to extremely low level in order to make the computation possible, which has been completed [10]. If surface tension exists, we can compute Bensimon's equation in the same way, but special method of numerical computation must be designed.

Pattern selections of nonsingular Saffman-Taylor finger in the presence of surface tension and multi-fingers constitute our further work.

References

1. Saffman P.G.& G.I.Taylor, Proc. Roy. Soc., London Ser. A, 245, 312(1958).
2. H.J.S.Hele-shaw, Nature, 58, 34(1898).
3. McLean J.W.& P.G.Saffman, J. Fluid Mech., 102, 455(1981).
4. M.Vanden-Brock, Phys. Fluids, 26, 2033(1983).
5. Dynamics of curved Fronts, edited by P. Pelce (Academic press, Inc., Boston, 1988).Asymptotic Beyond All Orders, edited by H. Segur et al., NATO ASI, Ser. B. ,Vol 284(Plenum Press, New York, 1991).B.I. Shraiman, Phys. Rev. Lett.,56, 2028(1986).
6. D. C. Hong and J. S. Langer, Phys. Rev. Lett., 56 ,2032(1986); R. Combescot et al., Phys. Rev. Lett., 56 2036(1986). S. Tanveer, Phys. Fluids , 30, 1589(1987).
7. Mark Mineev-Weinstein, Phys. Rev. Lett., 80 , 2113(1998).
8. D.Bensimon, Phys. Rev. A, 33, 1302(1986).
9. L.V.Ahlfors, Complex Analysis (McGraw-Hill, New York, 1953).
10. Huida Yu & Zhen Han, Phys. Rev. E, 58(5), 6873(1998).